Generalizing π, Angle Measure, and Trigonometry

Rob Poodiack

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1 Introduction

Pop quiz: What is the area enclosed by the graph of $|x|+|y| = 1$ (the unit circle under the "taxicab" metric) in Figure 1 below?

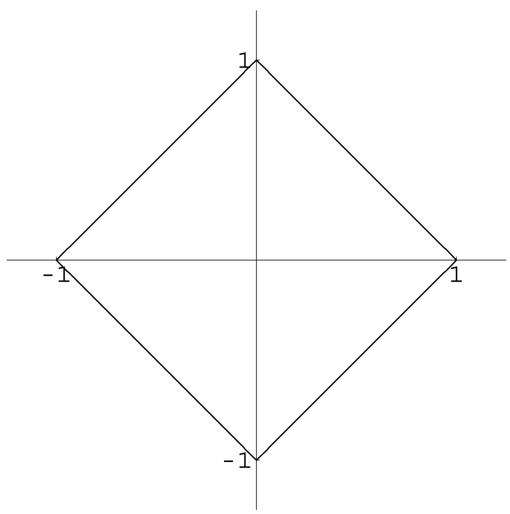

Figure 1: The graph of $|x| + |y| = 1$, the "taxicab" unit circle

Pencils up! How many of you said "2 square units"? A lot of people probably agreed with you. Without thinking too hard, you might have looked at the piece of the circle in the first quadrant, realized that we have half of a unit square, computed the area as $1/2$ and multiplied by four.

The length of the line segment (or arc of the taxicab circle) between $(1,0)$ and $(0,1)$ is $\sqrt{2}$ units in our usual Euclidean metric. Recall, though, that in the taxicab metric,

$$d((x_1, y_1), (x_2, y_2)) = |x_2 - x_1| + |y_2 - y_1|$$

Thus, in the taxicab metric, the length of our line segment is now 2 units, and the area of our taxicab unit circle is 4 square units.

Two recent articles in MAA journals ([1], [4]) proposed generalizations of the value of π if we equip $\mathbb{R}^2$ with non-Euclidean metrics. In particular, the authors asked what would happen if we replaced the usual distance formula by the more generic Minkowski ℓ^p metric

$$d_p((x_1, y_1), (x_2, y_2)) = \sqrt[p]{|x_2 - x_1|^p + |y_2 - y_1|^p}, p \geq 1.$$

The answer was that the value of π – the ratio of the circumference of the unit circle under the p-metric to its diameter – changed with p. (Let's follow these authors' lead and call this new value π_p.)

Thus π_1 would be the value of π under the taxicab metric. If we want to have the property that the area of the unit circle under the p-metric is π_p, what should we do?

Should $\pi_1 = 2$ or $\pi_1 = 4$?

Do we have similar discrepancies in the value of π for different p-metrics?

2 Formulas for π_p

There are some "desirable qualities" that $\pi = \pi_2$ possesses with respect to the unit circle:

1. π is the ratio of the circumference of the unit circle to its diameter.

2. π is the area (in square units) of the unit circle.

3. π is the arclength of the upper (or lower) half circle.

When generalizing, we'd like to take along as many of these properties as possible. In [1] and [4], the authors reasonably start with desirable qualities (1) and especially (3) – the arclength of the first quadrant of the unit circle, under the ℓ^p metric, should be $\pi_p/2$. Following [4], let's parametrize the unit circle as $x = u$ and $y = (1 - u^p)^{1/p}$, $0 \le u \le 1$. Using the p-arclength element,

$$ds_p = \sqrt[p]{\left|\frac{dx}{du}\right|^p + \left|\frac{dy}{du}\right|^p}\, du = \sqrt[p]{1 + \left(\frac{1}{u^p} - 1\right)^{1-p}}\, du$$

define

$$\pi_p = 2\int_0^1 \sqrt[p]{1 + \left(\frac{1}{u^p} - 1\right)^{1-p}}\, du$$

From this we have that $\pi_1 = 4$, $\pi_2 = \pi$, and $\pi_\infty = \lim_{p\to\infty} \pi_p = 4$. (In [1], Adler and Tanton use a different parametrization that gives the same values of π_p.) In fact the range of π_p is $[\pi, 4]$.

Forty-five years ago in [7], though, Shelupsky proposed values of π_p based on generalizations of the sine and cosine functions. Beginning with differential equations and solving, Shelupsky arrives at the formula (using Π_p for Shelupsky's version):

$$\Pi_p = 2\int_0^1 \frac{1}{(1-u^p)^{(p-1)/p}}\,du$$

In this case, we get $\Pi_1 = 2$, $\Pi_2 = \pi$ and $\Pi_\infty = 4$. (See Figure 2.)

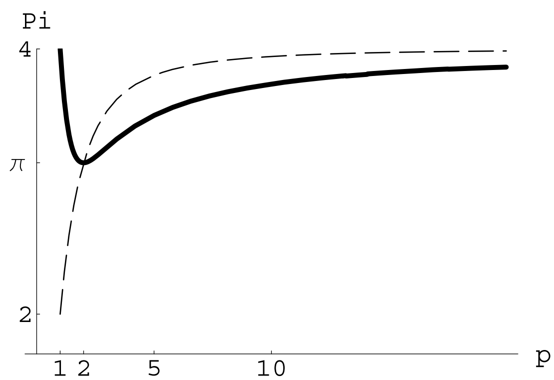

Figure 2: The graphs of π_p (black) and Π_p (dashed)

(Interestingly, in Lindqvist and Peetre's problem in the American Mathematical Monthly [5], they use Shelupsky's definition for the generalized sine and cosine functions. However, in his solution to this problem [2], Borwein uses Shelupsky's definition of what we're calling Π_p with a factor of p in front of the integral rather than a 2. This has the effect that Borwein's general $\Pi_p \approx 2p$ for large p.)

Shelupsky's generalization results in a range for Π_p of $[2, 4]$. If this seems a little strange, it might be that we're recalling a recent excellent article in this Journal [3]. Duncan, Luecking, and McGregor proved that the possible range for values of generalizations of π under *any* metric on $\mathbb{R}^2$ is $[3, 4]$ as long as π_p has desirable qualities (1) and (3). In fact, if the unit circle has quarter-turn symmetry, as do the circles under the ℓ^p metric, the range is $[\pi, 4]$.

So what gives with Shelupsky's definition? It depends on which of our three desirable qualities of π we want to maintain.

Recall that the Euclidean unit circle has the property that its area is $\pi_2 = \pi$ square units. In fact,

$$\pi = 4\int_0^1 \sqrt{1-x^2}\,dx$$

Shelupsky proves that

$$\Pi_p = 4\int_0^1 \sqrt[p]{1-x^p}\,dx$$

which four times the usual area under the curve $y = \sqrt[p]{1-x^p}$.

Aha! Shelupsky's generalizations all have the property that the area of the ℓ^p metric's unit circle is Π_p. For the other authors [1, 3, 4], π_p serves as the ratio of circumference to diameter as well as the arclength of the upper half circle.

It seems we can either have desirable qualities (1) and (3) or we can have (2), but not both. Actually, if we have (1) and (3), we can get (2) if we measure area in a different way than usual – recall our taxicab unit circle. But if we start with quality (2), we can't get (1) and (3). (The variety of ways area can be computed for the unit disc are discussed in [6].)

3 Trigonometric functions for π_p

If we do things right, we can happily end up with an angle measurement system such that the unit circle under the ℓ^p metric has $2\pi_p$ radians and the area of this circle is π_p square units.

To do this, we can't use our usual radian. We'll need a new variation on this unit for each $p \geq 1$. Let's develop some analogues of radian measure and the trigonometric functions for the ℓ^p metric. We first adapt some terminology from [8]:

Definition 3.1 *The* **unit p-circle** *is the graph of the equation* $|x|^p + |y|^p = 1$.

Definition 3.2 *A* **p-radian** *is an angle whose vertex is the center of a unit p-circle and intercepts an arc of p-length 1. The p-measure of an angle φ_p is the number of p-radians subtended by the angle on the unit p-circle about the origin.*

Since we defined π_p to be twice the p-arclength of the first-quadrant portion of the unit p-circle, it follows that a unit p-circle has $2\pi_p$ p-radians. Because of the symmetry of the unit p-circle, the Euclidean angles $\pi/4$, $\pi/2$ and π in standard position will have p-radian measure $\pi_p/4$, $\pi_p/2$, and π_p, respectively. (The best we'll be able to say of the rest is that, in general, $\alpha\pi$ radians corresponds to *about* $\alpha\pi_p$ p-radians in standard position.)

Now and in the sequel, we will use φ_p to refer to an angle in the unit p-circle (in p-radians), and θ to mean a Euclidean angle (in radians).

We would like to define new sine and cosine functions – let's use $S_p(\cdot)$ and $C_p(\cdot)$, respectively, for notation – as the usual y- and x-coordinates on the unit p-circle corresponding to an angle in p-radians.

Let's start with an acute Euclidean angle θ in standard position. Its terminal side has equation $y = (\tan\theta)x$. We wish to figure out the coordinates (x_0, y_0) of the point where this line intersects the unit p-circle. In the first quadrant, this circle has equation $y = (1 - x^p)^{1/p}$.

Setting $x = x_0$ and $y = y_0$, and solving, we obtain

$$x_0 = \frac{1}{(1 + \tan^p\theta)^{1/p}} \qquad y_0 = \left(1 - \frac{1}{1 + \tan^p\theta}\right)^{1/p} = \frac{\tan\theta}{(1 + \tan^p\theta)^{1/p}}$$

We know that x_0 should go with the new cosine function, and y_0 should go with the new sine function; in turn these functions should depend on an angle in p-radians, not just on the Euclidean angle θ. We now connect φ_p to θ.

Just as in the Euclidean metric, we define φ_p to be the p-arclength swept out by the Euclidean angle θ (in standard position) along the unit p-circle.

Definition 3.3 *Let $0 \leq \theta \leq \pi/2$ be a Euclidean angle. The corresponding angle φ_p in p-radians is given by*

$$\varphi_p(\theta) = \int_0^{S_p(\varphi_p(\theta))} \sqrt[p]{1 + \left(\frac{1}{u^p} - 1\right)^{1-p}}\, du$$

$$= \int_{C_p(\varphi_p(\theta))}^{1} \sqrt[p]{1 + \left(\frac{1}{u^p} - 1\right)^{1-p}}\, du \quad (*)$$

where

$$C_p(\varphi_p(\theta)) = x_0 = \frac{1}{(1 + \tan^p \theta)^{1/p}} = \frac{\cos\theta}{(\cos^p \theta + \sin^p \theta)^{1/p}}$$

$$S_p(\varphi_p(\theta)) = y_0 = \frac{\tan\theta}{(1 + \tan^p \theta)^{1/p}} = \frac{\sin\theta}{(\cos^p \theta + \sin^p \theta)^{1/p}}$$

As defined, we have $\varphi_p(0) = 0$, $\varphi_p(\pi/4) = \pi_p/4$, and $\varphi_p(\pi/2) = \pi_p/2$ for all $p \geq 1$. This is a result of the quarter-turn symmetry of the unit p-circle. (But note that we don't have any "finer" rotational symmetry unless $p = 2$.)

Again, the best we can say in general is that $\varphi_p(\alpha\pi) \approx \alpha\pi_p$ for $\alpha \in [0, 1/2]$.

While θ increases from 0 to $\pi/2$, $\varphi_p(\theta)$ increases from 0 to $\pi_p/2$. By symmetry of the unit p-circle, we may extend the domain of φ_p to the entire real line:

$$\varphi_p(\theta) = \varphi_p(\theta - \tfrac{\pi}{2}) + \tfrac{\pi_p}{2} \qquad \tfrac{\pi}{2} < \theta \leq \pi$$

$$\varphi_p(\theta) = \varphi_p(\theta - \pi) + \pi_p \qquad \pi < \theta \leq 2\pi$$

$$\varphi_p(\theta + 2\pi k) = \varphi_p(\theta) + 2\pi_p k \qquad \text{for all } k \in \mathbb{Z}$$

By a similar method, we can extend the domains of C_p and S_p in the same manner as for the usual sine and cosine:

$$C_p(\varphi_p) = -C_p(\pi_p - \varphi_p) \qquad S_p(\pi_p - \varphi_p) \qquad \frac{\pi_p}{2} \leq \varphi_p \leq \pi_p$$

$$C_p(\varphi_p) = C_p(2\pi_p - \varphi_p) \qquad S_p(\varphi_p) = -S_p(2\pi_p - \varphi_p) \qquad \pi_p < \varphi_p \leq 2\pi_p$$

$$C_p(\varphi_p + 2\pi_p k) = C_p(\varphi_p) \qquad S_p(\varphi_p + 2\pi_p k) = S_p(\varphi_p) \qquad \text{for all } k \in \mathbb{Z}$$

Thus S_p and C_p have period $2\pi_p$. A graph of one period of both C_{10} and S_{10} can be found in Figure 3. Note that while the graphs are similar to the usual sine and cosine graphs, the values do differ.

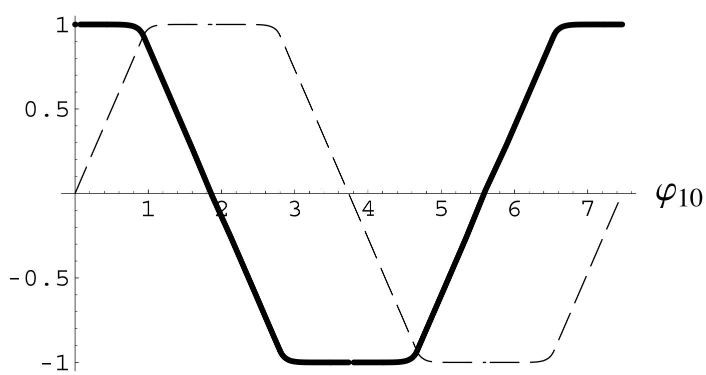

Figure 3: The graphs of $C_{10}(\varphi_{10})$ (black) and $S_{10}(\varphi_{10})$ (dashed). Note that $\pi_{10} \approx 3.73599$.

The definitions of the two functions and of φ_p lead immediately to the following properties for our new sine and cosine functions:

$$S_p^p(\varphi_p) + C_p^p(\varphi_p) = 1 \quad \text{for all } \varphi_p \in \mathbb{R}$$

$$S_p(\tfrac{\pi_p}{2} - \varphi_p) = C_p(\varphi_p) \qquad C_p(\tfrac{\pi_p}{2} - \varphi_p) = S_p(\varphi_p)$$

$$S_p(\varphi_p(0)) = S_p(0) = 0 \qquad S_p(\varphi_p(\tfrac{\pi}{2})) = S_p(\tfrac{\pi_p}{2}) = 1$$

$$C_p(\varphi_p(\tfrac{\pi}{4})) = C_p(\tfrac{\pi_p}{4}) = \tfrac{1}{\sqrt[p]{2}} \qquad S_p(\varphi_p(\tfrac{\pi}{4})) = S_p(\tfrac{\pi_p}{4}) = \tfrac{1}{\sqrt[p]{2}}$$

$$C_p(\varphi_p(0)) = C_p(0) = 1 \qquad C_p(\varphi_p(\tfrac{\pi}{2})) = C_p(\tfrac{\pi_p}{2}) = 0$$

$C_p(\varphi_p)$ is an even function of φ_p.

$S_p(\varphi_p)$ is an odd function of φ_p.

We now extend two results from [8]. (See Corollary 2.4 and Lemma 2.5.) When $p = 2$, angles have the same measure regardless of whether or not they are in standard position. For all other p, unfortunately, that is not the case; a Euclidean angle θ will generally not sweep out the same number of p-radians if it is rotated by ψ radians.

Theorem 3.4 *If an acute Euclidean angle θ is rotated by ψ radians away from standard position (that is, ψ is the reference angle for θ), then the number of p-radians swept out by the angle θ is*

$$\varphi_p(\theta + \psi) - \varphi_p(\psi) = \int_{S_p(\varphi_p(\psi))}^{S_p(\varphi_p(\theta+\psi))} \sqrt[p]{1 + \left(\frac{1}{u^p} - 1\right)^{1-p}}\, du$$

A "proof without words" can be seen in Figure 4.

Since $\varphi_p(\theta + \psi) - \varphi_p(\psi) \neq \varphi_p(\theta)$ in general, this means that angle measure is not rotationally invariant unless $p = 2$. One amazing fact, however, is that Euclidean right angles always correspond to right angles under p-radian measure.

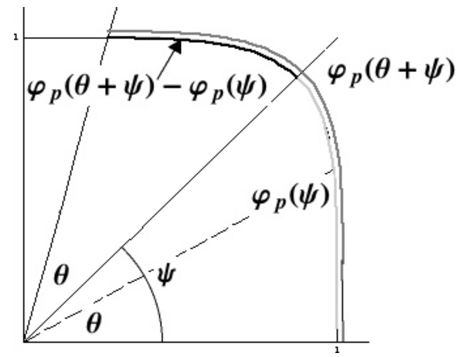

Figure 4: If the Euclidean angle θ is rotated by the Euclidean reference angle ψ, then the p-arclength swept out is $\varphi_p(\theta+\psi)-\varphi_p(\psi)$.

Theorem 3.5 *Euclidean right angles have a measure of $\pi_p/2$ p-radians.*

Proof. We follow the proof of [8], Lemma 2.5. Without loss of generality, let α_p be an angle surrounding the positive y-axis. We split α_p into two Euclidean angles by the positive y-axis, θ_1 the first quadrant piece, and θ_2, the second quadrant piece. Angle θ_1 has reference angle $\frac{\pi}{2}-\theta_1$ and θ_2 has reference angle $\frac{\pi}{2}-\theta_2$. By Theorem 3.4 and the fact that $\theta_1+\theta_2=\pi/2$, we get

$$\begin{aligned}\alpha_p &= \int_{S_p(\varphi_p(\frac{\pi}{2}-\theta_1))}^{S_p(\varphi_p(\frac{\pi}{2}))} \sqrt[p]{1+\left(\frac{1}{u^p}-1\right)^{1-p}}\,du + \int_{S_p(\varphi_p(\frac{\pi}{2}-\theta_2))}^{S_p(\varphi_p(\frac{\pi}{2}))} \sqrt[p]{1+\left(\frac{1}{u^p}-1\right)^{1-p}}\,du \\ &= \int_{C_p(\varphi_p(\theta_1))}^{1} \sqrt[p]{1+\left(\frac{1}{u^p}-1\right)^{1-p}}\,du + \int_{S_p(\varphi_p(\theta_1))}^{1} \sqrt[p]{1+\left(\frac{1}{u^p}-1\right)^{1-p}}\,du\end{aligned}$$

In the second integral, if we make the change of variables $t=(1-u^p)^{1/p}$, and use the fact that $C_p(\varphi_p(\theta_1))=(1-S_p^p(\varphi_p(\theta_1)))^{1/p}$, we get

$$\begin{aligned}\alpha_p &= \int_{C_p(\varphi_p(\theta_1))}^{1} \sqrt[p]{1+\left(\frac{1}{u^p}-1\right)^{1-p}}\,du + \int_{0}^{C_p(\varphi_p(\theta_1))} \sqrt[p]{1+\left(\frac{1}{t^p}-1\right)^{1-p}}\,dt \\ &= \int_{0}^{1} \sqrt[p]{1+\left(\frac{1}{u^p}-1\right)^{1-p}}\,du = \frac{\pi_p}{2}\end{aligned}$$

4 Derivatives and inverses

We look finally at the derivatives and inverses of our general sine and cosine functions. If we differentiate the first equation in (*) with respect to φ_p, we get:

$$\varphi_p = \int_0^{S_p(\varphi_p)} \sqrt[p]{1 + \left(\frac{1}{u^p} - 1\right)^{1-p}}\, du \Longrightarrow 1 = \sqrt[p]{1 + \left(\frac{1}{S_p^p(\varphi_p)} - 1\right)^{1-p}} \cdot S_p'(\varphi_p)$$

$$\Longrightarrow S_p'(\varphi_p) = \frac{1}{\sqrt[p]{1 + \left(\frac{1-S_p^p(\varphi_p)}{S_p^p(\varphi_p)}\right)^{1-p}}} = \frac{1}{\sqrt[p]{1 + \left(\frac{C_p^p(\varphi_p)}{S_p^p(\varphi_p)}\right)^{1-p}}}$$

If we define $T_p(\varphi_p) = S_p(\varphi_p)/C_p(\varphi_p)$, we can rewrite the above as

$$\Longrightarrow S_p'(\varphi_p) = \frac{1}{\left[1 + T_p^{p(p-1)}(\varphi_p)\right]^{1/p}} = \frac{C_p^{p-1}(\varphi_p)}{\left[C_p^{p(p-1)}(\varphi_p) + S_p^{p(p-1)}(\varphi_p)\right]^{1/p}}$$

Now if we differentiate the equation $S_p^p(\varphi_p) + C_p^p(\varphi_p) = 1$, substitute the above for $S_p'(\varphi_p)$ and solve for $C_p'(\varphi_p)$, we get the following formulas:

$$\boxed{\begin{aligned} S_p'(\varphi_p) &= \frac{C_p^{p-1}(\varphi_p)}{\left[C_p^{p(p-1)}(\varphi_p) + S_p^{p(p-1)}(\varphi_p)\right]^{1/p}} \\ C_p'(\varphi_p) &= \frac{-S_p^{p-1}(\varphi_p)}{\left[C_p^{p(p-1)}(\varphi_p) + S_p^{p(p-1)}(\varphi_p)\right]^{1/p}} \end{aligned}}$$

In Figure 5, we see the graph of C_4 along with that of its derivative.

Admittedly, these aren't as tidy as Shelupsky's formulas. But he *started* with the assumption that $S_p'(\varphi_p) = C_p^{p-1}(\varphi_p)$ and $C_p'(\varphi_p) = -S_p^{p-1}(\varphi_p)$ and got his definitions of S_p and C_p as well as his formula for Π_p from there. We note that our formulas do reduce to the standard sine and cosine differentiation formulas when $p = 2$.

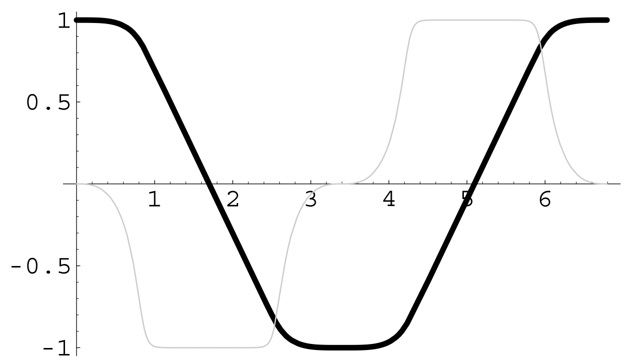

Figure 5: The graphs of $C_4(\varphi_4)$ (in black) and $C_4'(\varphi_4)$ (in gray).

The formulas in (*) also give convenient definitions for inverses for our general sine and cosine functions. For $-1 \leq x \leq 1$, we have

$$\boxed{\begin{aligned} S_p^{-1}(x) &= \int_0^x \sqrt[p]{1 + \left(\frac{1}{u^p} - 1\right)^{1-p}}\, du \\ C_p^{-1}(x) &= \int_x^1 \sqrt[p]{1 + \left(\frac{1}{u^p} - 1\right)^{1-p}}\, du \end{aligned}}$$

In particular, we have $S_p^{-1}(0) = 0$, $S_p^{-1}(1) = \pi_p/2$, $C_p^{-1}(1) = 0$, and $C_p^{-1}(0) = \pi_p/2$. From here, it is simple to get the derivatives of the inverse functions.

$$\boxed{\frac{d}{dx} S_p^{-1}(x) = -\frac{d}{dx} C_p^{-1}(x) = \sqrt[p]{1 + \left(\frac{1}{x^p} - 1\right)^{1-p}}}$$

5 A word about the tangent function

We defined our new tangent function awhile back but only used it as a means to an end. It has a fairly easy definition in terms of θ.

$$T_p(\varphi_p(\theta)) = \frac{S_p(\varphi_p(\theta))}{C_p(\varphi_p(\theta))} = \tan\theta$$

Because of the way we defined $\varphi_p(\theta)$, this shouldn't be a surprise: the line through the origin corresponding to $\varphi_p(\theta)$ p-radians has the same slope as the line through the origin corresponding to θ radians. Also,

$$1 + T_p^p(\varphi_p) = \frac{1}{C_p^p(\varphi_p)} \qquad \text{for all } \varphi_p \in \mathbb{R}$$

Can we come up with a defining equation for $T_p(\varphi_p)$ similar to the equations in (*)? By the Quotient Rule,

$$T_p'(\varphi_p) = \frac{S_p'(\varphi_p)C_p(\varphi_p) - C_p'(\varphi_p)S_p(\varphi_p)}{C_p^2(\varphi_p)}$$

By our previous formulas we have

$$\begin{aligned} S_p'(\varphi_p)C_p(\varphi_p) - C_p'(\varphi_p)S_p(\varphi_p) &= \frac{C_p^{p-1}(\varphi_p) \cdot C_p(\varphi_p) - (-S_p(\varphi_p))S_p^{p-1}(\varphi_p)}{\left[C_p^{p(p-1)}(\varphi_p) + S_p^{p(p-1)}(\varphi_p)\right]^{1/p}} \\ &= \frac{1}{\left[C_p^{p(p-1)}(\varphi_p) + S_p^{p(p-1)}(\varphi_p)\right]^{1/p}} \end{aligned}$$

Since $C_p(\varphi_p) = 1/(1 + T_p^p(\varphi_p))^{1/p}$, we have

$$T_p'(\varphi_p) = \frac{1}{C_p^2(\varphi_p)\left[C_p^{p(p-1)}(\varphi_p) + S_p^{p(p-1)}(\varphi_p)\right]^{1/p}}$$

$$= \frac{1}{\left(C_p^p(\varphi_p)\right)^{2/p}\left[C_p^{p(p-1)}(\varphi_p) + S_p^{p(p-1)}(\varphi_p)\right]^{1/p}}$$

$$= \frac{(1 + T_p^p(\varphi_p))^{2/p}}{(1 + T_p^{p(p-1)}(\varphi_p))^{1/p} \cdot C_p^{p-1}(\varphi_p)}$$

$$= \frac{(1 + T_p^p(\varphi_p))^{1/p+1}}{(1 + T_p^{p(p-1)}(\varphi_p))^{1/p}}$$

This means that

$$\frac{(1 + T_p^{p(p-1)}(\varphi_p))^{1/p}}{(1 + T_p^p(\varphi_p))^{1/p+1}} \cdot T_p'(\varphi_p) = 1$$

We can solve this differential equation via integration. Since we have the initial condition that $T_p(0) = 0$, we get

$$\boxed{\varphi_p(\theta) = \int_0^{T_p(\varphi_p(\theta))} \frac{\left(1 + t^{p(p-1)}\right)^{1/p}}{(1 + t^p)^{1/p+1}}\, dt}$$

We may then define the inverse of our new tangent function. For any real x, we have

$$\boxed{\begin{aligned} T_p^{-1}(x) &= \int_0^x \frac{\left(1 + t^{p(p-1)}\right)^{1/p}}{(1 + t^p)^{1/p+1}}\, dt \\ \frac{d}{dx} T_p^{-1}(x) &= \frac{\left(1 + x^{p(p-1)}\right)^{1/p}}{(1 + x^p)^{1/p+1}} \end{aligned}}$$

6 Conclusion

If we generalize π in the ℓ^p metric based on which desirable qualities we want, we get different values and very different trigonometric functions for these new π_p. It seems that only if $p = 2$ can we have our π and eat it too.

7 Question for Further Research

Is there an easy addition or subtraction formula for these generalized trigonometric functions?

References

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