# NES/MAA Collegiate Mathematics Competition 2008 Solutions

1. Find the exact value of
$$\int_0^1 \frac{4x^3(1+x^{4(2007)})}{(1+x^4)^{2009}}\,dx.$$

*Solution.* We show the more general statement that
$$\int_0^1 \frac{4x^3(1+x^{4(n-1)})}{(1+x^4)^{n+1}}\,dx = \frac{1}{n},$$
for positive integers $n$. Let $u = 1 + x^4$. Then $du = 4x^3\,dx$ and $x^4 = u - 1$ and we obtain
$$\begin{aligned}\int_0^1 \frac{4x^3(1+x^{4(n-1)})}{(1+x^4)^{n+1}}\,dx &= \int_1^2 \frac{1+(u-1)^{n-1}}{u^{n+1}}\,du \\ &= \int_1^2 u^{-n-1}\,du + \int_1^2 \frac{1}{u^2}\left(\frac{u-1}{u}\right)^{n-1}\,du \\ &= \left[\frac{u^{-n}}{-n}\right]_1^2 + \int_1^2 \frac{1}{u^2}\left(1-\frac{1}{u}\right)^{n-1}\,du\end{aligned}$$
In the remaining integral, we set $t = 1 - \frac{1}{u}$. Then $dt = \frac{1}{u^2}\,du$ and the desired integral is
$$\begin{aligned}&= \left(-\frac{1}{n2^n} + \frac{1}{n}\right) + \int_0^{1/2} t^{n-1}\,dt \\ &= \left(-\frac{1}{n2^n} + \frac{1}{n}\right) + \left[\frac{t^n}{n}\right]_0^{1/2} = \frac{1}{n}.\end{aligned}$$
Now just substitute $n = 2008$ to obtain the solution
$$\int_0^1 \frac{4x^3(1+x^{4(2007)})}{(1+x^4)^{2009}}\,dx = \frac{1}{2008}.$$

2. A boy goes off to college and after the first semester he's run out of money. In fact, he's so broke he doesn't have enough money to call home, or to even send a letter. However, he manages to find a postcard with a stamp already on it.

He writes the following message:

$$\begin{array}{cccccc} & & S & E & N & D \\ + & & M & O & R & E \\ \hline & M & O & N & E & Y \end{array}$$

Assuming his parents are able to decipher the code (and they don't tell him to drop dead), how much money did they send the boy?

*Solution.* We are adding two 4-digit numbers and getting a 5-digit number. This means that the $M$ in the word "MONEY" must be the result of a carry from adding $S + M$. Therefore $M = 1$.

We now know that the value of $S + M$ must be 10 or greater, otherwise there's no need for a carry. Since $M = 1$, this means $S = 9$ or possibly $S = 8$ (if there's a carry from the sum $E + O$).

Since $S = 8$ or $S = 9$, the sum $S + M$ adds up to 9, 10, or 11, depending on whether we carry from $E + O$. We rule out $S + M = 9$ since we know $S + M \geq 10$. So $O$ is 0 or 1. It can't be 1 because we already have that $M = 1$. Thus $O = 0$.

Suppose we try $S = 8$. Then our equation is:

$$\begin{array}{cccccc} & & 8 & E & N & D \\ + & & 1 & 0 & R & E \\ \hline & 1 & 0 & N & E & Y \end{array}$$

This would force $E = 9$ and we would have a carry from the sum $N + R$ to make $N = 0$ (and this would give a carry into the $S + M$ column). But we already have $O = 0$. So this doesn't work and we have $S = 9$.

Because $O = 0$ and $E \neq N$, we know that $E$ has to be exactly one less than $N$ (with a carry coming over from the $N + R$ column). Playing with consecutive number pairs for $E$ and $N$ eventually leads us to the combination $E = 5$ and $N = 6$. (The pairings $(E, N) = (2, 3), (3, 4)$ and $(4, 5)$ do not work because of the $E$ in the last column.) This gives us

$$\begin{array}{cccccc} & & 9 & 5 & 6 & D \\ + & & 1 & 0 & R & 5 \\ \hline & 1 & 0 & 6 & 5 & Y \end{array}$$

We now see that $R = 8$ or $R = 9$. Since $S = 9$, we have that $R = 8$.

This means we need a carry from the last column, implying that $D \geq 5$. Since we've assigned 5, 6, 8, and 9 already, this forces $D = 7$ and $Y = 2$.

Our full equation is thus

$$\begin{array}{cccccc} & & 9 & 5 & 6 & 7 \\ + & & 1 & 0 & 8 & 5 \\ \hline & 1 & 0 & 6 & 5 & 2 \end{array}$$

The boy's parents sent him \$10,652. (I'm trading in my parents!)

3. Solve the equation for $x$ in terms of $c$:

$$2 \log_x c - \log_{cx} c - 3 \log_{c^2 x} c = 0.$$

*Solution.* If $y = \log_a x$ then $a^y = x \implies a = x^{1/y} \implies \frac{1}{y} = \log_x a$. That is, $\log_a x$ and $\log_x a$ are reciprocals. So we rewrite the given equation as

$$\frac{2}{\log_c x} - \frac{1}{\log_c cx} - \frac{3}{\log_c c^2 x} = 0.$$

Set $k = \log_c x$. Then $\log_c cx = \log_c x + \log_c c = k + 1$ and $\log_c c^2 x = \log_c x + 2 \log_c c = k + 2$. So the equation becomes:

$$\begin{aligned} \frac{2}{k} - \frac{1}{k+1} - \frac{3}{k+2} = 0 &\implies 2(k+1)(k+2) - k(k+2) - 3k(k+1) = 0 \\ &\implies 2(k^2 + 3k + 2) - (k^2 + 2k) - 3(k^2 + k) = 0 \\ &\implies -2k^2 + k + 4 = 0 \implies 2k^2 - k - 4 = 0 \\ &\implies k = \frac{1 \pm \sqrt{33}}{4}. \end{aligned}$$

Therefore, $x = c^k = c^{(1 \pm \sqrt{33})/4}$.

4. The increasing sequence $S = \{2, 3, 5, 6, 7, 10, 11, \dots\}$ consists of all positive integers which are neither a perfect square nor a perfect cube. What is the 500th term of $S$?

*Solution.* Clearly the desired term $n$ is unique and greater than 500. Since the squares and cubes thin out as we go, $n$ is not likely to be much larger than 500.

The number of squares less than 500 is 22 ($22^2 = 484$) and the number of cubes is 7 ($7^3 = 343$ but $8^3 = 512$). Thus, up to 500, no more than 29 numbers fail to be in the set $S$. In fact, the numbers 1 and 64 are both perfect squares and perfect cubes. Therefore, only 27 integers fail membership in $S$, making 500 the 473rd member of $S$. Counting forward 27 entries brings us to 528 since only 512 fails to be in $S$. The 500th term of $S$ is 528.

*Note:* This cuts it close because $529 = 23^2$ is the next integer we drop.

5. Alice, Bob, and Carol repeatedly take turns tossing a fair regular six-sided die. Alice begins; Bob always follows Alice; Carol always follows Bob; and Alice always follows Carol. Find the probability that Carol will be the first to toss a six.

   *Solution.* If Carol wins in the first round, then she must have rolled a six after two non-sixes have occurred. This happens with probability

$$\frac{5}{6} \cdot \frac{5}{6} \cdot \frac{1}{6} = \frac{1}{6}\left(\frac{5}{6}\right)^2.$$

   If Carol wins in the second round, five non-sixes preceded her lucky six. This happens with probability

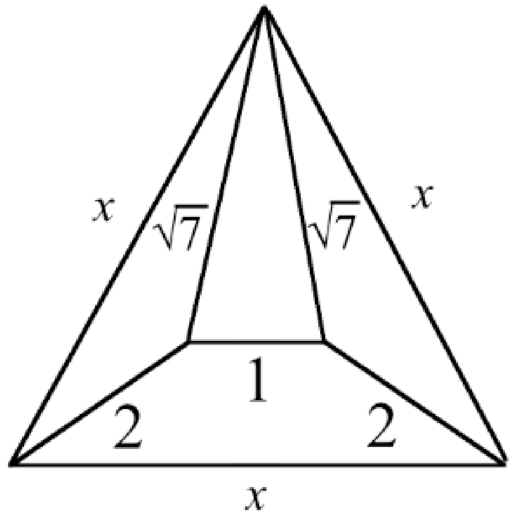
$$\frac{5}{6} \cdot \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{1}{6} = \frac{1}{6}\left(\frac{5}{6}\right)^5.$$

   This pattern continues. Carol wins in the third round with probability $\frac{1}{6}\left(\frac{5}{6}\right)^8$ and in the fourth with probability $\frac{1}{6}\left(\frac{5}{6}\right)^{11}$. It *is* possible (though unlikely) that the game could continue forever.

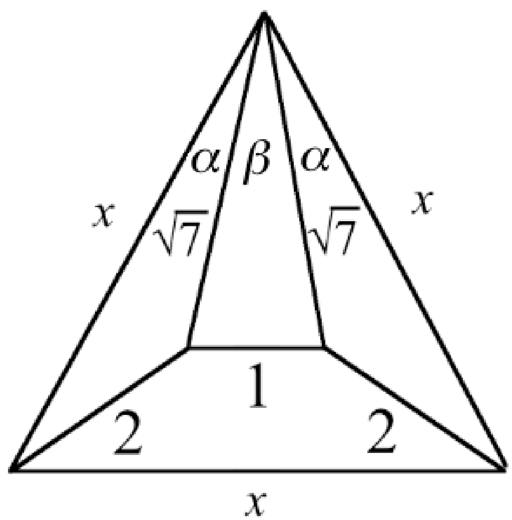
   The probability that Carol wins the game is equal to the sum of the probabilities that she wins in any given round:

$$\begin{aligned}
\frac{1}{6}\left(\frac{5}{6}\right)^2 + \frac{1}{6}\left(\frac{5}{6}\right)^5 + \frac{1}{6}\left(\frac{5}{6}\right)^8 + \cdots &= \frac{1}{6}\left(\frac{5}{6}\right)^2\left[1 + \left(\frac{5}{6}\right)^3 + \left(\frac{5}{6}\right)^6 + \cdots\right] \\
&= \frac{1}{6}\left(\frac{5}{6}\right)^2 \sum_{n=0}^{\infty}\left(\frac{5}{6}\right)^{3n} \\
&= \frac{1}{6}\left(\frac{5}{6}\right)^2 \cdot \frac{1}{1-(5/6)^3} \\
&= \frac{1}{6}\left(\frac{5}{6}\right)^2 \cdot \frac{1}{1-125/216} \\
&= \frac{1}{6}\left(\frac{5}{6}\right)^2 \cdot \frac{1}{91/216} \\
&= \frac{1}{6} \cdot \frac{25}{36} \cdot \frac{216}{91} \\
&= \frac{25}{91}.
\end{aligned}$$

6. Determine $x$ in the following equilateral triangle:




*Solution.* Consider the figure below.




By the Law of Cosines on the center triangle,

$$\begin{aligned} 1^2 = (\sqrt{7})^2 + (\sqrt{7})^2 - 2 \cdot \sqrt{7} \cdot \sqrt{7} \cos\beta &\Longrightarrow 1 = 7 + 7 - 14\cos\beta \\ &\Longrightarrow \cos\beta = \tfrac{13}{14}. \end{aligned}$$

Thus $\sin\beta = \sqrt{1-\cos^2\beta} = \sqrt{1 - \frac{169}{196}} = \sqrt{27/196} = \frac{\sqrt{27}}{14}$. From here, we can try to figure out some things about angle $\alpha$ and then use the Law of Cosines a second time to get the value of $x$.

We know that since the big triangle is equilateral that $2\alpha + \beta = \pi/3$. Therefore,

$$\begin{aligned} \cos 2\alpha &= \cos\left(\tfrac{\pi}{3} - \beta\right) \\ &= \cos\tfrac{\pi}{3}\cos\beta + \sin\tfrac{\pi}{3}\sin\beta \\ &= \frac{1}{2} \cdot \frac{13}{14} + \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{27}}{14} \\ &= \frac{13}{28} + \frac{9}{28} = \frac{11}{14}. \end{aligned}$$

This leads to

$$\cos^2\alpha = \frac{1+\cos 2\alpha}{2} = \frac{1+\frac{11}{14}}{2} = \frac{25}{28},$$

and $\cos\alpha = \frac{5}{2\sqrt{7}}$. If we use the Law of Cosines on, say, the left triangle, we have

$$2^2 = x^2 + (\sqrt{7})^2 - 2x\sqrt{7}\cos\alpha = x^2 + 7 - 2x\sqrt{7}\cdot\frac{5}{2\sqrt{7}} = x^2 + 7 - 5x.$$

So $x^2 - 5x + 3 = 0 \Longrightarrow x = \frac{1}{2}(5 \pm \sqrt{13})$. Because $x > 1$, this gives us $x = \frac{1}{2}(5 + \sqrt{13})$.

7. Snow begins to fall during the morning of November 21 and continues steadily into the afternoon. At noon a snowplow begins removing snow from a road at a constant rate. The plow travels 6 km from noon to 1 p.m. but only 3 km from 1 p.m. to 2 p.m. To the nearest minute, when did the snow begin to fall? [*Hint:* Let $t$ be the time measured in hours after noon; let $x(t)$ be the distance traveled by the plow at time $t$. Then the speed of the plow is $dx/dt$. Let $b$ be the number of hours before noon that it began to snow. Find an expression for the height of the snow at time $t$. Then use the given information that the rate of removal $R$ (in $\text{m}^3/\text{h}$) is constant.]

   *Solution.* Let $b$, $t$ and $x = x(t)$ be as defined in the hint. Since the snow falls steadily – at a constant rate $k$, say – the height of the snow at time $t$ is given by $h(t) = k(t+b)$.

   We are given that the rate of removal $R$ is constant. If the width of the path is $w$ then,

$$R = \text{height} \times \text{width} \times \text{speed} = h(t) \times w \times \frac{dx}{dt} = k(t+b)w\,\frac{dx}{dt}.$$

   Thus, $\frac{dx}{dt} = \frac{R}{kw(t+b)} \equiv \frac{C}{t+b}$, where $C = R/(kw)$ is a constant. This is a separable differential equation.

$$\frac{dx}{dt} = \frac{C}{t+b} \Longrightarrow \int dx = C\int \frac{dt}{t+b} \Longrightarrow x(t) = C\ln(t+b) + K.$$

   Now to deal with the constants.

   - Set $t = 0$:

$$0 = C\ln b + K \Longrightarrow K = -C\ln b \Longrightarrow x(t) = C\ln(t+b) - C\ln b = C\ln\left(1 + \frac{t}{b}\right).$$

   - Set $t = 1$: Since $R$ is measured in $\text{m}^3/\text{h}$, this means that $x$ is measured in meters. After 1 hour, the plow has traveled 6 km = 6000 m. So $x(1) = 6000 = C\ln(1 + \frac{1}{b})$.
   - Set $t = 2$. After 2 hours, the plow has traveled 9 km = 9000 m and $x(2) = 9000 = C\ln(1 + \frac{2}{b})$.

   Given this, we can eliminate $C$ and solve for $b$:

$$\begin{aligned} C = \frac{6000}{\ln(1+\frac{1}{b})} = \frac{9000}{\ln(1+\frac{2}{b})} &\Longrightarrow 3\ln\left(1+\frac{1}{b}\right) = 2\ln\left(1+\frac{2}{b}\right) \\ &\Longrightarrow \left(1+\frac{1}{b}\right)^3 = \left(1+\frac{2}{b}\right)^2 \\ &\Longrightarrow 1 + \frac{3}{b} + \frac{3}{b^2} + \frac{1}{b^3} = 1 + \frac{4}{b} + \frac{4}{b^2} \\ &\Longrightarrow \frac{1}{b} + \frac{1}{b^2} - \frac{1}{b^3} = 0 \\ &\Longrightarrow b^2 + b - 1 = 0 \Longrightarrow b = \frac{-1 \pm \sqrt{5}}{2} \end{aligned}$$

   But $b > 0$, so $b = \frac{-1+\sqrt{5}}{2} \approx 0.618$ h $\approx 37$ min. The snow began to fall at about 11:23 a.m.