NES/MAA Collegiate Mathematics Competition 2007 Solutions

1. Calculate exactly (*i.e.*, no decimals) $\lim_{x\to 3} \frac{x}{x-3}\left(\int_3^x \sin t\, dt\right)$.

   Solution. If we substitute $x = 3$ into the various pieces of the given function, we get the indeterminate $\frac{0}{0}$, which gets us nowhere. Instead, let $F(x) = \int_3^x \sin t\, dt$. Then $F(3) = 0$ and

$$\begin{aligned}\lim_{x\to 3} \frac{x}{x-3}\left(\int_3^x \sin t\, dt\right) &= \lim_{x\to 3} x \cdot \lim_{x\to 3} \frac{F(x)}{x-3} \\ &= 3 \cdot \lim_{x\to 3} \frac{F(x)-F(3)}{x-3} \\ &= 3\cdot F'(3) \\ &= 3\sin 3,\end{aligned}$$

   by the definition of the derivative and the Fundamental Theorem of Calculus.

2. Determine the value of

$$S = \log(\tan 1^\circ) + \log(\tan 2^\circ) + \cdots + \log(\tan 88^\circ) + \log(\tan 89^\circ).$$

   Solution. We note that for any $0^\circ < x < 90^\circ$

$$\tan(90^\circ - x) = \frac{\sin(90^\circ - x)}{\cos(90^\circ - x)} = \frac{\cos x}{\sin x} = \cot x$$

   Therefore, $\tan x \cdot \tan(90^\circ - x) = 1$ for any $0^\circ < x < 90^\circ$ and we can write

$$\begin{aligned}S &= \log(\tan 1^\circ) + \log(\tan 2^\circ) + \cdots + \log(\tan 88^\circ) + \log(\tan 89^\circ) \\ &= \log(\tan 1^\circ \cdot \tan 2^\circ \cdots \tan 88^\circ \cdot \tan 89^\circ) \\ &= \log\left((\tan 1^\circ \cdot \tan 89^\circ)(\tan 2^\circ \cdot \tan 88^\circ)\cdots(\tan 44^\circ \cdot \tan 46^\circ)\cdot \tan 45^\circ\right) \\ &= \log(1\cdot 1 \cdots 1 \cdot \tan 45^\circ) \\ &= \log(\tan 45^\circ).\end{aligned}$$

   So $S = \log(\tan 45^\circ) = \log 1 = 0$.

3. Experiments have determined that when a particular steel ball is bounced on a hard surface, it bounces to half its original height. For example, if it is dropped from a height of 6 feet, it will bounce to 3 feet. Assuming that the ball obeys this law exactly, for what length of time will the ball continue to bounce if it is dropped from a height of 16 feet? (Or will it bounce forever?)

   [Recall from calculus that since the acceleration due to gravity is 32 ft/sec^2, an object falling to the ground from height h (in feet) or bouncing from the ground to height h requires $\sqrt{h}/4$ seconds to do so.]

   Solution. If the ball obeys the law exactly, it must bounce an infinite number of times, but it is not clear whether it bounces for an infinite time.

   Since the ball is dropped from a height of 16 feet, it will require $\sqrt{16}/4 = 1$ second to fall to the ground. It then bounces to a height of 8 feet, requiring $\sqrt{8}/4$ seconds to rise and $\sqrt{8}/4$ seconds to fall to the ground again. Continuing in this manner, we see that the total time is the sum of the series

$$1 + 2(\sqrt{8}/4) + 2(\sqrt{4}/4) + 2(\sqrt{2}/4) + \cdots,$$

or

$$1+\frac{2}{4}\sum_{k=0}^{\infty}\sqrt{\frac{8}{2^k}}=1+\sqrt{2}\sum_{k=0}^{\infty}\left(\frac{1}{\sqrt{2}}\right)^k.$$

This is a geometric series with common ratio $r=1/\sqrt{2}<1$. Thus the series converges and its sum is

$$1+\sqrt{2}\cdot\frac{1}{1-1/\sqrt{2}}=\frac{\sqrt{2}+1}{\sqrt{2}-1}.$$

Thus this model predicts the ball will bounce slightly more than 5.8 seconds.

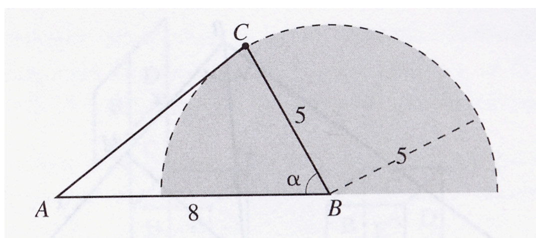
4. Prove that a group G of order 15 must be cyclic.

   Solution. The order of an element divides the order of G, and can therefore be 1, 3, 5, or 15. For these k, let n_k be the number of elements of order k. Clearly $n_1=1$. We will show that $n_{15}>0$, which will mean that an element x of order 15 exists and so $G=\langle x\rangle$ is cyclic.

   Suppose G has a subgroup(s) of order 3 and b subgroup(s) of order 5. Since 3 and 5 are prime, by Sylow's theorem, $a\equiv 1 \pmod 3$, $b\equiv 1 \pmod 5$, and both a and b divide 15. Thus $a=b=1$ and $n_3=2$, $n_5=4$. Since $n_1+n_3+n_5+n_{15}=15$, $n_{15}=15-1-2-4=8>0$.

5. An object moves 8 cm in a straight line from A to B, turns at an angle α, measured in radians and chosen at random from the interval $(0,\pi)$, and moves 5 cm in a straight line to C. What is the probability that $AC<7$?

   Solution. The point C will lie on the semicircle of radius 5 shown in the figure below.



   First we find the value of α which gives $AC=7$. The Law of Cosines applied to $\triangle ABC$ implies that

$$7^2=5^2+8^2-2\cdot 5\cdot 8\cos\alpha, \quad \text{so} \quad \cos\alpha=\tfrac{1}{80}(25+64-49)=\tfrac{1}{2},$$

   and $\alpha=\pi/3$. Thus for $\alpha<\pi/3$ we have $AC<7$ and when $\alpha\geq\pi/3$ we have $AC\geq 7$. So the probability that $AC<7$ is $(\pi/3)/\pi=1/3$.

6. A and B are positive integers in the decimal system such that

   - $A=7B$ and
   - the sum of the digits of A is twice the sum of the digits of B.

   If C is the number formed by writing the digits of B immediately after the digits of A, prove that C is a multiple of 9.

   Solution. If B has k digits, then

$$\begin{aligned} C &= (\ldots\text{digits of } A\ldots)(\ldots\text{the } k \text{ digits of } B\ldots)\\ &= A\cdot 10^k+B\\ &= 7B\cdot 10^k+B\\ &= B(7\cdot 10^k+1).\end{aligned}$$

Let $S(n)$ denote the sum of the digits of the integer n. It is given that $S(A) = 2S(B)$. Therefore, $S(C) = S(A) + S(B) = 3S(B)$.

Since $S(C)$ is divisible by 3, so is C. That is, $3 \mid B(7 \cdot 10^k + 1)$. But

$$S(7 \cdot 10^k + 1) = S(700 \dots 01) = 8,$$

implying that $7 \cdot 10^k + 1$ is not divisible by 3, and so it must be that $3 \mid B$. Thus $3 \mid S(B)$, making $S(C) = 3 \cdot S(B) = 3(3t) = 9t$ for some positive integer t. Since 9 divides $S(C)$, 9 divides C also.

7. A group of 11 scientists wants to design a cabinet in which to keep their top secret papers. They propose to outfit the cabinet with a set of locks and supply a certain number of keys to each scientist so that the cabinet can be opened only when a majority of the scientists is present. That is,

   - no group of 5 or fewer is to be able to open all the locks, and
   - every group of 6 or more is to have keys for all the locks.

   What is the minimum number of locks that must be built into the cabinet and what is the minimum number of keys each scientist must be given?

   Solution. For each group of five scientists, there must be at least one lock to which none has the key, and for any two groups, these locks must be different (else the addition to one group of a new scientist from the other would provide a majority no member of which contains a key for such a lock as we have associated with the first group). Therefore, we need as many locks as there are groups of five scientists, namely

$$\binom{11}{5} = 462 \text{ different locks.}$$

   Since each scientist must carry a key for each subset of 5 other scientists (which his presence converts from a minority to a majority, thus requiring him to supply the key to the lock they can't open), each scientist would be obliged to carry around

$$\binom{10}{5} = 252 \text{ different keys.}$$

   Thus some ideas that sound plausible are quite impractical!