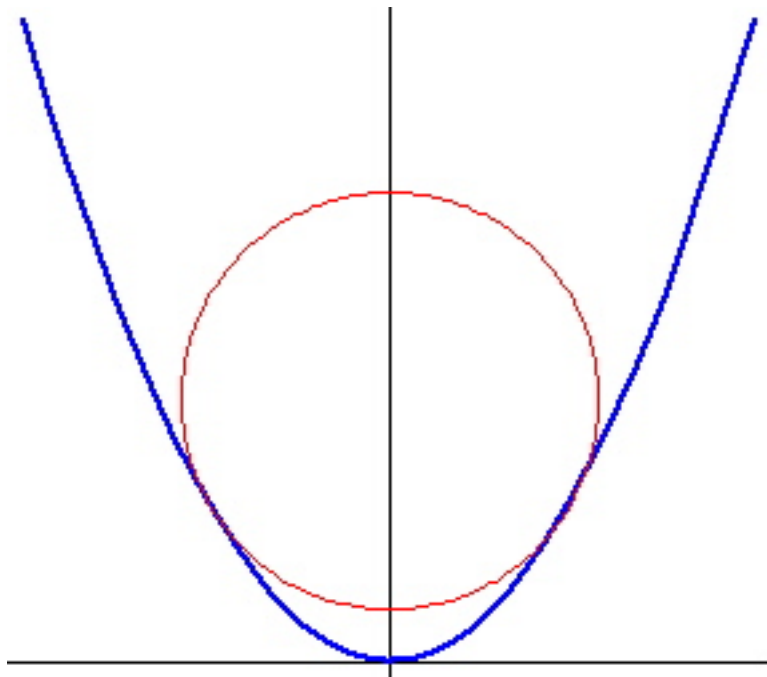




NES/MAA Collegiate Mathematics Competition 2006 Solutions

1. The figure below shows a circle with radius 1 inscribed in the parabola $y = x^2$. Find the center of the circle.



Solution. Let the center of the circle be $(0, a)$. Then the equation of the circle is $x^2 + (y - a)^2 = 1$. For the points that are also on the parabola, $y = x^2$. Substituting into the circle equation for y , we have $x^2 + (x^2 - a)^2 = 1$. This leads to

$$\begin{aligned}x^2 + (x^2 - a)^2 = 1 &\iff x^2 + x^4 - 2ax^2 + a^2 = 1 \\ &\iff x^4 + (1 - 2a)x^2 + (a^2 - 1) = 0.\end{aligned}$$

The parabola and circle will be tangent to each other precisely when this equation (in x^2) has a double root; that is, when the discriminant is 0. Thus

$$\begin{aligned}(1 - 2a)^2 - 4(1)(a^2 - 1) = 0 &\iff 1 - 4a + 4a^2 - 4a^2 + 4 = 0 \\ &\iff -4a = -5 \\ &\iff a = \frac{5}{4}.\end{aligned}$$

The center of the circle is $(0, \frac{5}{4})$.

2. Which number is bigger: e^π or π^e ? (Your calculator will of course tell you the answer; you need to prove it.)

Solution. Given two numbers a^b and b^a ($a > 0, b > 0$), suppose $a^b < b^a$. Then we would have

$$\begin{aligned}a^b < b^a &\iff \ln(a^b) < \ln(b^a) \\ &\iff b \ln a < a \ln b \\ &\iff \frac{\ln a}{a} < \frac{\ln b}{b}.\end{aligned}$$

This suggests we look at the function $f(x) = \frac{\ln x}{x}$. Then $f'(x) = (1 - \ln x)/x^2$, and $f'(x) = 0$ when $\ln x = 1$; that is, when $x = e$. Since f' changes sign from positive to negative at $x = e$, f has a relative maximum at $x = e$. It is also an absolute maximum since $f(x) \rightarrow 0$ as $x \rightarrow \infty$ and $f(x) \rightarrow -\infty$ as $x \rightarrow 0^+$.

Thus $f(\pi) < f(e)$, so $\ln \pi / \pi < \ln e / e$, which implies $\pi^e < e^\pi$.

3. If x , y , and z are positive numbers, prove that

$$\frac{(x^2 + 1)(y^2 + 1)(z^2 + 1)}{xyz} \geq 8.$$

Solution. Let $f(x) = \frac{x^2+1}{x} = x + \frac{1}{x}$. We note that $f'(x) = 1 - \frac{1}{x^2} = \frac{x^2-1}{x^2}$. So f has a critical point at $x = 1$ (it has others, but not at positive x). Since f' changes from negative to positive as it passes $x = 1$, we have a local minimum at $x = 1$; in fact we have an absolute minimum there since $f(x) \rightarrow \infty$ as $x \rightarrow 0^+$ and as $x \rightarrow \infty$.

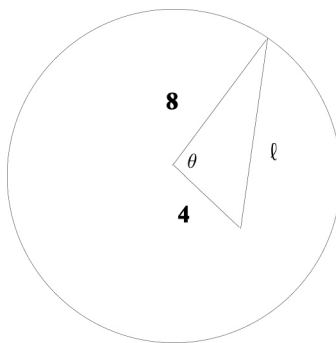
So $f(x) \geq (1 + \frac{1}{1}) = 2$ for all positive x . Therefore,

$$\begin{aligned} \frac{(x^2 + 1)(y^2 + 1)(z^2 + 1)}{xyz} &= \left(\frac{x^2 + 1}{x}\right) \left(\frac{y^2 + 1}{y}\right) \left(\frac{z^2 + 1}{z}\right) \\ &= f(x) \cdot f(y) \cdot f(z) \\ &\geq 2 \cdot 2 \cdot 2 = 8. \end{aligned}$$

(The AM-GM inequality could be used here as well.)

4. The minute hand on a watch is 8 mm long and the hour hand is 4 mm long. How fast is the distance between the tips of the hands changing at one o'clock?

Solution. Here is a diagram to help us.



The hour hand of a clock goes around once every 12 hours or, in radians per hour, $\frac{2\pi}{12} = \frac{\pi}{6}$ rad/h.

The minute hand goes around once per hour, or at the rate of 2π rad/h.

So the angle θ between them (measuring clockwise from the minute hand to the hour hand) is changing at the rate of

$$\frac{d\theta}{dt} = \frac{\pi}{6} - 2\pi = -\frac{11\pi}{6} \text{ rad/h.}$$

To relate θ to ℓ , we use the Law of Cosines:

$$\ell^2 = 8^2 + 4^2 - 2 \cdot 8 \cdot 4 \cos \theta = 80 - 64 \cos \theta. \tag{*}$$

We differentiate (*) implicitly with respect to t :

$$2\ell \frac{d\ell}{dt} = -64(-\sin \theta) \frac{d\theta}{dt}. \tag{**}$$

At 1:00, the angle between them is one-twelfth of the circle, that is $\theta = \frac{2\pi}{12} = \frac{\pi}{6}$ radians. We can now use (*) to find ℓ :

$$\ell = \sqrt{80 - 64 \cos \frac{\pi}{6}} = \sqrt{80 - 32\sqrt{3}}.$$

Substituting this into (**), we get

$$2\ell \frac{d\ell}{dt} = 64 \sin \frac{\pi}{6} \cdot \left(-\frac{11\pi}{6}\right) \implies \frac{d\ell}{dt} = \frac{64 \left(\frac{1}{2}\right) \left(-\frac{11\pi}{6}\right)}{2\sqrt{80 - 32\sqrt{3}}} = -\frac{88\pi}{\sqrt{80 - 32\sqrt{3}}} \approx -18.6$$

So the distance between the tips of the hands is decreasing at a rate of -18.6 mm/h at 1:00.

5. Let A be a square matrix and suppose that there exist positive integers m and n such that $A^m = I$ and $A^n \neq I$. Calculate

$$\det(I + A + A^2 + \cdots + A^{m-1}).$$

Solution. The standard formula for a partial sum of a geometric series has a matrix analogue:

$$(I + A + A^2 + \cdots + A^{m-1})(I - A) = (I - A^m).$$

We have that $I - A^m = O$ by hypothesis; therefore,

$$(I + A + A^2 + \cdots + A^{m-1})(I - A)\mathbf{x} = \mathbf{0} \tag{***}$$

no matter what the vector \mathbf{x} . Since $A \neq I$, there exist nonzero vectors \mathbf{x} and \mathbf{y} such that

$$(I - A)\mathbf{x} = \mathbf{y}.$$

But by (***) we have

$$(I + A + A^2 + \cdots + A^{m-1})\mathbf{y} = \mathbf{0},$$

so that $I + A + A^2 + \cdots + A^{m-1}$ is singular and $\det(I + A + A^2 + \cdots + A^{m-1}) = 0$.

6. Can a group be a union of two proper subgroups?

Solution. Suppose that $G = H \cup K$ where H and K are proper subgroups of G . Then there exists an element $h \in H$ but $h \notin K$ and an element $k \in K$ but $k \notin H$.

Let $g = hk$. Certainly $g \in G$ and so either $g \in H$ or $g \in K$. If $g \in H$, then so is $k = h^{-1}g$, a contradiction. This would force $g \in K$, but then $h = gk^{-1} \in K$, another contradiction.

So a group cannot be a union of two proper subgroups.

7. A chicken and a half can lay an egg and a half in a day and a half. How long will it take for two chickens to lay 32 eggs?

Solution. We would have that 3 chickens can lay 3 eggs in a day and a half, and so 3 chickens can lay 6 eggs in 3 days. This would mean that 2 chickens can lay 4 eggs in 3 days, and thus **2 chickens can lay 32 eggs in 24 days.**